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On the unsteady Stokes problem with a nonlinear open artificial boundary condition modelling a singular load

Philippe Angot

Aix-Marseille Université, Laboratoire d'Analyse, Topologie, Probabilités - CNRS UMR7353, Centre de Mathématiques et Informatique, 13453 Marseille Cedex 13 - France.

Abstract

We propose a practical nonlinear open boundary condition of Robin type for unsteady incompressible viscous flows taking account of the local inflow/outflow volume rate at an open artificial boundary with a singular load. The inflow/outflow parameters introduced in the modelling can be connected to the coefficient of singular head loss through Bernouilli's theorem of energy balance in a curl-free viscous flow. Then, we prove that this boundary condition leads to a well-posed unsteady nonlinear Stokes problem, *i.e.* global in time existence of a weak solution in dimension $d \leq 3$ with no restriction on the data. The proof is carried out by passing to the limit on a sequence of consistent discrete solutions of a non linear numerical scheme which approximates the original problem. The main ingredients are Schauder's fixed-point theorem and Aubin-Lions compactness argument.

Keywords: Open boundary condition, inflow/outflow boundary condition, artificial boundary, singular head loss, non stationary Stokes problem, unsteady incompressible viscous flow.
2010 MSC: 35A35, 35J50, 35J60, 35J65, 35K40, 35K55, 35K60, 35Q30, 76D03, 76D05, 65M06, 65M12, 65N12

1. Introduction

Let Ω be a bounded and connected open set of $\mathbb{R}^d$, for $d = 2$ or 3, with a Lipschitz continuous boundary $\Gamma = \partial\Omega$ and ν be the outward unit normal vector on Γ. We study the non stationary Stokes problem associated with the following non linear Robin boundary condition for the traction on the whole boundary Γ:

$$\begin{cases} \dfrac{\partial v}{\partial t} - \mathrm{div}\,(2\mu\, D(v)) + \nabla p = f \quad \text{in } \Omega \times (0,T), \\ \mathrm{div}\, v = 0 \quad \text{in } \Omega \times (0,T), \\ v(t=0) = v_0 \quad \text{in } \Omega. \end{cases} \tag{1}$$

$$(\mathrm{OBC}_{|\Gamma}) \left\{ \sigma(v,p)\cdot\nu + \frac{1}{2}\alpha(v\cdot\nu)\, v = 0 \quad \text{on } \Gamma\times(0,T), \quad \text{with} \qquad \alpha(v\cdot\nu) = \alpha_-(v\cdot\nu)^- + \alpha_+(v\cdot\nu)^+. \right. \tag{2}$$

We recall that the positive and negative parts of any real number $x \in \mathbb{R}$ are defined by

$$x^+ = \max(x,0), \quad x^- = -\min(x,0) = \max(-x,0) = (-x)^+, \quad x = x^+ - x^-, \quad |x| = \max(x,-x) = x^+ + x^-.$$

Here, the tensor $D(v)$ denotes the symmetic part of ∇v and $\sigma(v,p)\cdot\nu = -p\,\nu + 2\mu\, D(v)\cdot\nu$ is the stress vector on Γ, the viscosity μ satisfying $0 < \mu_m \leq \mu(x) \leq \mu_M$ a.e. in Ω. The given dimensionless parameters $0 \leq \alpha_-, \alpha_+ \leq 1$ represent the local rates of kinetic energy transfer for the inflow/outflow on Γ due to a singular load downstream from the artificial boundary. So, by the generalized Bernouilli's theorem of energy balance in a curl-free incompressible viscous flow

Email address: `philippe.angot@univ-amu.fr` (**Philippe Angot**)
URL: `http://www.latp.univ-mrs.fr/~angot` (**Philippe Angot**)

[12], they are connected to the coefficient of singular head loss for which there exist many tables in experimental fluid mechanics; see [13, Chap. XII]. Indeed, a part of the kinetic energy on the artificial boundary can be locally absorbed (for outflow) or provided (for inflow) by the singular load and the other part of the head loss being controlled by the exterior pressure p_e included in the given traction force on Γ, for instance $g = -p_e\, v$, which is taken to zero here. By introducing these inflow/outflow parameters α_-, α_+, we generalize Bruneau & Fabrie's outflow boundary condition which needs a reference solution [10, 11]; see also [6, 22]. Moreover, by choosing $\alpha_- = \alpha_+ = 0$, the famous free outflow or *do nothing* condition, see e.g. [19, 20, 21, 2, 5], is still available although it only gives local existence in time or global existence with small data for Navier-Stokes problems; see [25, 20, 23] and also [14, 15] for related problems.

It is also possible to penalize the Robin condition by taking $\alpha(v \cdot \nu)_{|\Gamma_D} = 1/\varepsilon$ on a part Γ_D of Γ to reach a velocity Dirichlet condition at the limit when the penalty parameter $\varepsilon > 0$ tends to zero; see [21, 2, 7]. Finally, such a condition ($\mathrm{OBC}_{|\Gamma}$) can be a basis towards some fluid-porous transmission problems like in [1, 2, 3] or [4].

We use below the usual Sobolev functional setting and notations for Navier-Stokes equations; see [9, 26, 8]. Since we have a traction condition on the whole boundary Γ, we work in the Hilbert space $\mathbf{W}$ defined by

$$\mathbf{W} = \left\{ w \in \mathbf{H}^1(\Omega);\ \operatorname{div} w = 0 \ \text{ in } \ \Omega \right\}$$

equipped with the natural inner product and associated norm in $\mathbf{H}^1(\Omega)$. Then, we have the continuous imbeddings:

$$\begin{cases} (i) & \mathbf{W} \subset \mathbf{L}^2(\Omega) = \mathbf{L}^2(\Omega)' \subset \mathbf{W}' \\ (ii) & \mathbf{H}^{\frac{1}{2}}(\Gamma) \subset \mathbf{L}^2(\Gamma) = \mathbf{L}^2(\Gamma)' \subset \mathbf{H}^{-\frac{1}{2}}(\Gamma). \end{cases} \tag{3}$$

We use throughout the paper the notation $c(\Omega)$ or $C(\Omega)$ to denote a generic positive "constant" depending only on Ω.

We associate to the Stokes problem (1,2) the weak problem below stated using usual integrations by part with Green's formula. For $v_0 \in \mathbf{L}^2(\Omega)$ and $f \in L^2\left(0, T; \mathbf{L}^2(\Omega)\right)$ given, find v in $L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; \mathbf{W})$ such that $\frac{dv}{dt} \in L^1(0, T; \mathbf{W}')$, $v(t = 0) = v_0$ and satisfying in $\mathcal{D}'(]0, T[)$:

$$\frac{d}{dt} \int_\Omega v \cdot \varphi \, dx + \int_\Omega 2\mu\, D(v) : D(\varphi) \, dx + \frac{1}{2} \int_\Gamma \alpha(v \cdot \nu)\, v \cdot \varphi \, ds = \int_\Omega f \cdot \varphi \, dx, \qquad \forall \varphi \in \mathbf{W}. \tag{4}$$

Then, we have the main result of this paper which is proved in several steps detailed in Sections 2 and 3.

Theorem 1.1 (Existence of weak solution to the nonlinear Stokes problem (1,2) for $d \leq 3$). *For all $v_0 \in \mathbf{L}^2(\Omega)$, $f \in L^2(0, T; \mathbf{L}^2(\Omega))$ and for $\alpha(.) \geq 0$ a.e. on Γ given, the problem (1,2) admits at least a weak solution (v, p) satisfying (4) with $v \in L^\infty(0, T; \mathbf{L}^2(\Omega)) \cap L^2(0, T; \mathbf{W})$, $\dfrac{dv}{dt} \in L^{\frac{4}{3}}(0, T; \mathbf{W}')$ and $p \in W^{-1,\infty}(0, T; L^2(\Omega))$. Moreover, v is weakly continuous from $[0, T]$ into $\mathbf{L}^2(\Omega)$ with $v(0) = v_0$ in $\mathbf{L}^2(\Omega)$ weakly and satisfies the energy inequality for all $t \in [0, T]$:*

$$\frac{1}{2} \int_\Omega |v(t)|^2 \, dx + \int_0^t \int_\Omega 2\mu\, |D(v)|^2 \, dx\, d\tau + \frac{1}{2} \int_0^t \int_\Gamma \alpha(v \cdot \nu)\, |v|^2 \, ds\, d\tau \leq \frac{1}{2} \int_\Omega |v_0|^2 \, dx + \int_0^t \int_\Omega f(\tau) \cdot v(\tau) \, dx\, d\tau. \tag{5}$$

2. Solvability of discrete Stokes problems with ($\mathrm{OBC}_{|\Gamma}$)

In this section, we prove the existence of weak solution to a non linear semi-discretization scheme in time which approximates the Stokes problem (1,2). The proof is carried out using Schauder's fixed-point theorem, e.g. [17], with some compactness argument.

For a given time step $0 < \delta t \leq T$, possibly sufficiently small, u^n denotes a desired approximation of any function $u(.)$ at the time $t_n = n\delta t$ for all $n \in \mathbb{N}$ such that $(n + 1)\delta t \leq T$. More precisely, we have for $f \in L^2(0, T; \mathbf{L}^2(\Omega))$:

$$f^{n+1} = \frac{1}{\delta t} \int_{n\delta t}^{(n+1)\delta t} f(t) \, dt \in \mathbf{L}^2(\Omega), \qquad \text{such that} \quad \sum_{k=0}^{n} \delta t \left\| f^{k+1} \right\|_{0,\Omega}^2 \leq \sum_{k=0}^{n} \int_{k\delta t}^{(k+1)\delta t} \left\| f(t) \right\|_{0,\Omega}^2 dt \leq \int_0^T \left\| f(t) \right\|_{0,\Omega}^2 dt.$$

Let us consider the backward-Euler semi-discretization of the problem (1,2). Here, the non linear term over the boundary is treated fully-implicitly in time. With the initial data $v^0 = v_0 \in \mathbf{L}^2(\Omega)$, the time discrete solution (v^{n+1}, p^{n+1}) satisfies the scheme below for all $n \in \mathbb{N}$ such that $(n+1)\delta t \leq T$:

$$\begin{cases} \dfrac{v^{n+1} - v^n}{\delta t} - \operatorname{div}(2\mu\, D(v^{n+1})) + \nabla p^{n+1} = f^{n+1} \quad \text{ in } \Omega \\ \operatorname{div} v^{n+1} = 0 \quad \text{ in } \Omega \\ \sigma(v^{n+1}, p^{n+1}) \cdot \nu + \dfrac{1}{2}\alpha(v^{n+1} \cdot \nu)\, v^{n+1} = 0 \quad \text{ on } \Gamma. \end{cases} \tag{6}$$

Then, the weak problem associated to the discrete scheme (6) and corresponding to the time discretization of the weak problem (4) reads as follows. For $v_0 \in \mathbf{L}^2(\Omega)$ and $f \in L^2(0,T;\mathbf{L}^2(\Omega))$ given, find (v^{n+1}, p^{n+1}) in $\mathbf{W} \times L^2(\Omega)$ with $v^0 = v_0$ and satisfying for all $n \in \mathbb{N}$ such that $(n+1)\delta t \leq T$:

$$\int_\Omega \frac{v^{n+1} - v^n}{\delta t} \cdot \varphi\, dx + \int_\Omega 2\mu\, D(v^{n+1}) : D(\varphi)\, dx + \frac{1}{2}\int_\Gamma \alpha(v^{n+1} \cdot \nu)\, v^{n+1} \cdot \varphi\, ds = \int_\Omega f^{n+1} \cdot \varphi\, dx, \qquad \forall \varphi \in \mathbf{W}. \tag{7}$$

2.1. *A priori energy estimates of discrete solutions*

Assuming that discrete solutions exist for the previous numerical scheme (6), at least with a time step δt small enough, we prove some energy bounds showing the unconditional stability of this scheme.

Theorem 2.1 (**Stability of the discrete scheme (6,7)**). *For $v_0 \in \mathbf{L}^2(\Omega)$, $f \in L^2(0,T;\mathbf{L}^2(\Omega))$ and for all $\alpha(.) \geq 0$ a.e. on Γ given, let us assume that there exists, at least for δt small enough, a time discrete solution (v^{n+1}, p^{n+1}) in $\mathbf{W} \times L^2(\Omega)$ to the numerical scheme (6,7). Then, there exists a bound $C_0 = C_0(\Omega, T, v_0, f) > 0$ depending only on the data such that, for all $0 < \delta t < \min(1,T)$, the following energy estimate holds for all $n \in \mathbb{N}$ such that $(n+1)\delta t \leq T$:*

$$\left\|v^{n+1}\right\|_{0,\Omega}^2 + \sum_{k=0}^{n} \left\|v^{k+1} - v^k\right\|_{0,\Omega}^2 + 4\sum_{k=0}^{n} \delta t \left\| \sqrt{\mu}\, D(v^{k+1})\right\|_{0,\Omega}^2 + \sum_{k=0}^{n} \delta t \left\| \sqrt{\alpha(v^{k+1} \cdot \nu)}\, v^{k+1}\right\|_{0,\Gamma}^2 \leq C_0(\Omega, T, v_0, f). \tag{8}$$

Proof. By choosing the test function $\varphi = 2\delta t\, v^{n+1} \in \mathbf{W}$ in (7), we get through classical calculations:

$$\left\|v^{n+1}\right\|_{0,\Omega}^2 - \left\|v^n\right\|_{0,\Omega}^2 + \left\|v^{n+1} - v^n\right\|_{0,\Omega}^2 + 4\delta t \left\| \sqrt{\mu}\, D(v^{n+1})\right\|_{0,\Omega}^2 + \delta t \left\| \sqrt{\alpha(v^{n+1} \cdot \nu)}\, v^{n+1}\right\|_{0,\Gamma}^2 = 2\delta t \int_\Omega f^{n+1} \cdot v^{n+1}\, dx.$$

Using the Cauchy-Schwarz and Young inequalities, the right-hand side term can be bounded as below:

$$2\delta t \int_\Omega f^{n+1} \cdot v^{n+1}\, dx \leq 2\delta t \left\|f^{n+1}\right\|_{0,\Omega} \left\|v^{n+1}\right\|_{0,\Omega} \leq \delta t \left\|f^{n+1}\right\|_{0,\Omega}^2 + \delta t \left\|v^{n+1}\right\|_{0,\Omega}^2.$$

Using this bound and then summing up, we get the desired estimate thanks to an implicit version of the discrete Gronwall inequality which holds for $\delta t < 1$ with $\sum_{k=0}^{n} \delta t \leq T$, together with the initial condition $v^0 = v_0$. □

2.2. *Existence result of the time discrete solution to the scheme (6,7)*

We begin by the following preliminary technical lemmas.

Lemma 2.2 (**Lipschitz continuity of** $\alpha(.)$). *For $0 \leq \alpha_-, \alpha_+ \leq 1$, let us define the positive function $\alpha : \mathbb{R} \longrightarrow \mathbb{R}^+$ by $\alpha(v \cdot \nu) = \alpha_-(v \cdot \nu)^- + \alpha_+(v \cdot \nu)^+$ for all $v \in \mathbb{R}^d$. Then, we have for all $u, v \in \mathbb{R}^d$:*

$$0 \leq \alpha(v \cdot \nu) \quad \leq \quad |v|, \qquad \text{and} \qquad |\alpha(u \cdot \nu) - \alpha(v \cdot \nu)| \leq |(u - v) \cdot \nu| \leq |u - v|\,,$$

where $|.|$ denotes the absolute value of a real number or the Euclidean norm in $\mathbb{R}^d$, as far as there is no confusion.

Proof. It is an easy matter using the properties of positive and negative parts of any real number and $|\nu| = 1$. □

Lemma 2.3 (**Estimates for the non linear boundary term for** $d \leq 3$)**.** *There exists $c(\Omega) > 0$ (different for each inequality) such that for all $u, v, w \in \mathbf{H}^1(\Omega)$, the estimates below hold for $d \leq 3$:*

$$(i) \quad \int_\Gamma |u|\, v \cdot w\, ds \leq c(\Omega) \left\| u \right\|_{1,\Omega} \left\| v \right\|_{1,\Omega} \left\| w \right\|_{0,\Omega}^{\frac{1}{2}} \left\| w \right\|_{1,\Omega}^{\frac{1}{2}}$$

$$(ii) \quad \int_\Gamma |u|\, v \cdot w\, ds \leq c(\Omega) \left\| u \right\|_{0,\Omega}^{\frac{1}{4}} \left\| u \right\|_{1,\Omega}^{\frac{3}{4}} \left\| v \right\|_{1,\Omega} \left\| w \right\|_{0,\Omega}^{\frac{1}{4}} \left\| w \right\|_{1,\Omega}^{\frac{3}{4}}.$$

Proof. We use Hölder inequalities combined with Sobolev imbeddings [9] as well as refined trace inequalities from [8, Lemma V.2.2]. For *(i)*, we take for instance $u \in \mathbf{L}^4(\Gamma)$, $v \in \mathbf{L}^4(\Gamma)$ and $w \in \mathbf{L}^2(\Gamma)$. For *(ii)*, we choose for example $u,\ w \in \mathbf{L}^{\frac{8}{3}}(\Gamma)$ and $v \in \mathbf{L}^4(\Gamma)$. Then, the results follow with trace and interpolation inequalities [8, Proposition II.3.7 & Lemma V.2.2]. □

Lemma 2.4 (**Compact set in** $\mathbf{L}^2(\Omega)$)**.** *For all $\delta t > 0$, $\mu_m > 0$ and $R > 0$, let us define the subset of $\mathbf{H}^1(\Omega)$*

$$\mathcal{K}_R(\Omega) = \left\{ w \in \mathbf{W},\ \left\| w \right\|_{0,\Omega}^2 \leq R \ \text{ and } \ \mu_m\, \delta t \left\| D(w) \right\|_{0,\Omega}^2 \leq R \right\}.$$

Then, $\mathcal{K}_R(\Omega) \subset \mathbf{W}$ is a compact and convex subset in $\mathbf{L}^2(\Omega)$.

Proof. The proof is immediate using Korn's first inequality [18, Chap. I.5] and the compact injection of $\mathbf{H}^1(\Omega)$ into $\mathbf{L}^2(\Omega)$ thanks to Rellich's compactness theorem, see e.g. [9]. □

For any $\overline{v}$ given in $\mathbf{H}^1(\Omega)$, let us first consider the following linear discrete scheme. For $v_0 \in \mathbf{L}^2(\Omega)$, $f \in L^2(0, T; \mathbf{L}^2(\Omega))$ given, find $(v^{n+1} = v(\overline{v}), p^{n+1} = p(\overline{v}))$ in $\mathbf{W} \times L^2(\Omega)$ with $v^0 = v_0$ and satisfying for all $n \in \mathbb{N}$ such that $(n+1)\delta t \leq T$:

$$\int_\Omega \frac{v^{n+1} - v^n}{\delta t} \cdot \varphi\, dx + \int_\Omega 2\mu\, D(v^{n+1}) : D(\varphi)\, dx + \frac{1}{2} \int_\Gamma \alpha(\overline{v} \cdot \nu)\, v^{n+1} \cdot \varphi\, ds = \int_\Omega f^{n+1} \cdot \varphi\, dx, \qquad \forall \varphi \in \mathbf{W}. \tag{9}$$

Then, we have the following well-posedness and energy estimate results for the linear problem.

Theorem 2.5 (**Existence and uniqueness of solution to discrete Stokes problem (9)**)**.** *Let $\overline{v}$ be given in $\mathbf{H}^1(\Omega)$. Then, with $v_0 \in \mathbf{L}^2(\Omega)$, $f \in L^2(0, T; \mathbf{L}^2(\Omega))$ and $\alpha(.) \geq 0$ a.e. on Γ given, and for all $\delta t > 0$, there exists for all $n \in \mathbb{N}$ such that $(n+1)\delta t \leq T$ a unique solution $(v^{n+1} = v(\overline{v}), p^{n+1}) = p(\overline{v})$ in $\mathbf{W} \times L^2(\Omega)$ to the discrete Stokes problem (9).*

Moreover, there exists a bound $C_0 = C_0(\Omega, T, v_0, f) > 0$ depending only on the data such that, for all $0 < \delta t < \min(1, T)$, the following energy estimate holds for all $n \in \mathbb{N}$ such that $(n+1)\delta t \leq T$:

$$\left\| v^{n+1} \right\|_{0,\Omega}^2 + \sum_{k=0}^{n} \left\| v^{k+1} - v^k \right\|_{0,\Omega}^2 + 4 \sum_{k=0}^{n} \delta t \left\| \sqrt{\mu}\, D(v^{k+1}) \right\|_{0,\Omega}^2 + \sum_{k=0}^{n} \delta t \left\| \sqrt{\alpha(\overline{v} \cdot \nu)}\, v^{k+1} \right\|_{0,\Gamma}^2 \leq C_0(\Omega, T, v_0, f). \tag{10}$$

Proof. The proof is made by an easy induction. We first apply at each time step $t_n = n\delta t$ the Lax-Milgram theorem [9] in the Hilbert space $\mathbf{W}$ to solve the weak problem (9) for $v^{n+1} \in \mathbf{W}$. The continuity of the bilinear form is obtained using Lemma 2.2 and *(i)* in Lemma 2.3 whereas the coercivity in $\mathbf{H}^1(\Omega)$ is ensured thanks to Korn's classical inequality and $\alpha(.) \geq 0$. Then, the estimate (10) holds similarly as (8) in Theorem 2.1 with the same bound C_0.

Conversely, by taking in (9) a smooth and compactly supported test function $\varphi \in C_c^\infty(\Omega)^d = \mathcal{D}(\Omega)^d$ such that $\operatorname{div} \varphi = 0$ in Ω, we get using De Rham's theorem in $\mathbf{H}^{-1}(\Omega)$ [18, Lemma I.2.1] that there exists $p_0^{n+1} \in L_0^2(\Omega)$, which is unique if Ω is connected, and such that the Stokes equation in (6) is satisfied in $\mathbf{H}^{-1}(\Omega)$. Then, using [18, Lemma I.2.2] we construct an *ad hoc* divergence-free extension in $\mathbf{W}$ of any function in $\mathbf{H}^{\frac{1}{2}}(\Gamma)$. Thanks to that with $\varphi \in \mathbf{W}$ in (9) and adapting the proof given in [8, Chap. III.5.2] for the Stokes problem with a Neumann boundary condition, it turns out that the open condition ($\text{OBC}_{|\Gamma}$) is satisfied in $\mathbf{H}^{-\frac{1}{2}}(\Gamma)$ with the pressure field, unique as soon as v^{n+1} and $\overline{v}$ are given, $p^{n+1}(\overline{v}) = p_0^{n+1} + C_\Gamma^{n+1}(\overline{v}) \in L^2(\Omega)$ where the constant $C_\Gamma^{n+1}(\overline{v})$ is given by, $|\Gamma|$ being the measure of Γ:

$$C_\Gamma^{n+1}(\overline{v}) = \frac{1}{|\Gamma|} \left\langle \left(\sigma(v^{n+1}, p_0^{n+1}) \cdot \nu + \frac{1}{2}\, \alpha(\overline{v} \cdot \nu)\, v^{n+1} \right), \nu \right\rangle_{-\frac{1}{2},\Gamma}.$$

□

Theorem 2.6 (Existence of solution to discrete Stokes scheme (6,7) for $d \le 3$). *Let us give $v_0 \in \mathbf{L}^2(\Omega)$, $f \in L^2(0,T;\mathbf{L}^2(\Omega))$, $\alpha(.) \ge 0$ a.e. on Γ and $0 < \delta t < \min(1,T)$.*

Then for all $n \in \mathbb{N}$ such that $(n+1)\delta t \le T$, there exists at least one solution (v^{n+1}, p^{n+1}) in $\mathbf{W} \times L^2(\Omega)$ to the non linear discrete scheme (6,7). Moreover, this solution satisfies the stability estimate (8) stated in Theorem 2.1.

Proof. Let us choose $R = C_0$ where $C_0 = C_0(\Omega, T, v_0, f) > 0$ is the bound given in (10) stated by Theorem 2.5. We consider the set $\mathcal{K}_R(\Omega) \subset \mathbf{W}$, as defined in Lemma 2.4, which is a convex and compact subset of $\mathbf{L}^2(\Omega)$. Let us now define the application $S : \mathcal{K}_R(\Omega) \longrightarrow \mathbf{W}$ by $S(\overline{v}) = v^{n+1}$ for all $\overline{v} \in \mathcal{K}_R(\Omega)$, where $v^{n+1} = v(\overline{v}) \in \mathbf{W}$ is the solution of the linear discrete Stokes problem (9) as stated in Theorem 2.5. First, we notice that the range of S, $S(\mathcal{K}_R(\Omega))$ with $R = C_0$, is included in $\mathcal{K}_R(\Omega)$. Indeed, whatever $\overline{v}$ in $\mathbf{H}^1(\Omega)$ and in particular for all $\overline{v} \in \mathcal{K}_R(\Omega) \subset \mathbf{W}$, the solution $v^{n+1} = v(\overline{v}) \in \mathbf{W}$ of (9) satisfies the energy estimate (10) given in Theorem 2.5. With $\mu(.) \ge \mu_m$, this implies that for all $n \in \mathbb{N}$ such that $(n+1)\delta t \le T$, $v^{n+1} = S(\overline{v})$ belongs to $\mathcal{K}_R(\Omega)$ with $R = C_0$ and we have $S(\mathcal{K}_R(\Omega)) \subset \mathcal{K}_R(\Omega)$.

Let us now prove that the application S is a continuous map on $\mathcal{K}_R(\Omega)$. For any $\overline{v} \in \mathcal{K}_R(\Omega)$, let us consider a sequence $(\overline{v}_l)_{l\in\mathbb{N}}$ in $\mathcal{K}_R(\Omega)$ which converges to $\overline{v}$, *i.e.* $\|\overline{v}_l - \overline{v}\|_{1,\Omega} \to 0$ when $l \to +\infty$. We define $w_l^{n+1} = S(\overline{v}_l) - S(\overline{v}) = v_l^{n+1} - v^{n+1} \in \mathbf{W}$ with $w_l^0 = 0$. Then, the weak problem satisfied by w_l^{n+1} for all $\varphi \in \mathbf{W}$ reads:

$$\int_\Omega \frac{w_l^{n+1} - w_l^n}{\delta t} \cdot \varphi \, dx + \int_\Omega 2\mu \, D(w_l^{n+1}) : D(\varphi)\, dx + \frac{1}{2}\int_\Gamma \alpha(\overline{v}_l \cdot \nu)\, w_l^{n+1} \cdot \varphi \, ds = -\frac{1}{2}\int_\Gamma (\alpha(\overline{v}_l \cdot \nu) - \alpha(\overline{v} \cdot \nu))\, v^{n+1} \cdot \varphi \, ds.$$

We now choose the test function $\varphi = 2\delta t\, w_l^{n+1} \in \mathbf{W}$ and bound the right-hand side term as below. For that, we use the Lipschitz continuity of $\alpha(.)$ from Lemma 2.2 and the estimate (i) in Lemma 2.3. Then, we get

$$\delta t \left| \int_\Gamma (\alpha(\overline{v}_l \cdot \nu) - \alpha(\overline{v} \cdot \nu))\, v^{n+1} \cdot w_l^{n+1} \, ds \right| \le c(\Omega)\, \delta t \, \|\overline{v}_l - \overline{v}\|_{1,\Omega} \|v^{n+1}\|_{1,\Omega} \|w_l^{n+1}\|_{1,\Omega}.$$

Using this bound and summing up similarly as in the proof of Theorem 2.1 with $w_l^0 = 0$, it yields:

$$\|w_l^{n+1}\|_{0,\Omega}^2 + \sum_{k=0}^n \|w_l^{k+1} - w_l^k\|_{0,\Omega}^2 + 4\sum_{k=0}^n \delta t \, \|\sqrt{\mu}\, D(w_l^{k+1})\|_{0,\Omega}^2 + \sum_{k=0}^n \delta t \, \|\sqrt{\alpha(\overline{v}_l \cdot \nu)}\, w_l^{k+1}\|_{0,\Gamma}^2$$
$$\le c(\Omega)\, \|\overline{v}_l - \overline{v}\|_{1,\Omega} \sum_{k=0}^n \delta t \, \|v^{k+1}\|_{1,\Omega} \|w_l^{k+1}\|_{1,\Omega}.$$

We now observe that the sequence $\|w_l^{k+1}\|_{1,\Omega} = \|v_l^{k+1} - v^{k+1}\|_{1,\Omega}$ is bounded independently of l since $v_l^{n+1} = S(\overline{v}_l)$ belongs to $\mathcal{K}_R(\Omega)$ with $R = C_0$ for all $n \in \mathbb{N}$ such that $(n+1)\delta t \le T$, whatever l. Then, passing to the limit for $l \to +\infty$ in the previous inequality, we get since $\|\overline{v}_l - \overline{v}\|_{1,\Omega} \to 0$ that $\|w_l^{n+1}\|_{1,\Omega} = \|S(\overline{v}_l) - S(\overline{v})\|_{1,\Omega} \to 0$. This shows that S is a continuous map from $\mathcal{K}_R(\Omega)$ onto $\mathcal{K}_R(\Omega)$.

Then, as a consequence of Schauder's fixed-point theorem, e.g. [17], the application S has a fixed point in $\mathcal{K}_R(\Omega)$, still denoted by $v^{n+1} \in \mathbf{W}$. Hence, for this fixed point with $\overline{v} = v^{n+1}$, the corresponding solution v^{n+1}, p^{n+1} in $\mathbf{W} \times L^2(\Omega)$ to the scheme (9) is a solution to the discrete Stokes problem (6,7) and satisfies the energy estimate (8) stated in Theorem 2.1. This concludes the proof. □

Lemma 2.7 (Bound for the time derivative for the discrete problem (6,7) with $d \le 3$). *For $v_0 \in \mathbf{L}^2(\Omega)$, $f \in L^2(0,T;\mathbf{L}^2(\Omega))$ and for all $\alpha(.) \ge 0$ a.e. on Γ given, let us consider a solution (v^{n+1}, p^{n+1}) in $\mathbf{W} \times L^2(\Omega)$ to the discrete Stokes problem (6,7) as stated in Theorem 2.6 with $0 < \delta t < \min(1,T)$. Then, there exists $C_1 = C_1(\Omega, T, \mu_M, v_0, f) > 0$ depending only on the data such that the following bound holds for all $n \in \mathbb{N}$ such that $(n+1)\delta t \le T$:*

$$\sum_{k=0}^n \delta t \left\| \frac{v^{k+1} - v^k}{\delta t} \right\|_{\mathbf{W}'}^{\frac{4}{3}} \le C_1(\Omega, T, \mu_M, v_0, f). \tag{11}$$

Proof. From the weak form (7) with (i) in (3), we have for all $\varphi \in \mathbf{W}$:

$$\left\langle \frac{v^{k+1} - v^k}{\delta t}, \varphi \right\rangle_{\mathbf{W}',\mathbf{W}} = \int_\Omega \frac{v^{k+1} - v^k}{\delta t} \cdot \varphi \, dx = \int_\Omega f^{k+1} \cdot \varphi \, dx - \int_\Omega 2\mu \, D(v^{k+1}) : D(\varphi)\, dx - \frac{1}{2}\int_\Gamma \alpha(v^{k+1} \cdot \nu)\, v^{k+1} \cdot \varphi \, ds.$$

Let us give a bound for each term in the right-hand side for $d = 3$.

$$\int_\Omega f^{k+1} \cdot \varphi \, dx \le \left\|f^{k+1}\right\|_{0,\Omega} \left\|\varphi\right\|_{0,\Omega} \le \left\|f^{k+1}\right\|_{0,\Omega} \left\|\varphi\right\|_{1,\Omega}.$$

$$\begin{aligned}\int_\Omega 2\mu \, D(v^{k+1}) : D(\varphi) \, dx &\le 2\mu_M \left\|D(v^{k+1})\right\|_{0,\Omega} \left\|D(\varphi)\right\|_{0,\Omega} \\ &\le 2\mu_M \left\|D(v^{k+1})\right\|_{0,\Omega} \left\|\varphi\right\|_{1,\Omega}.\end{aligned}$$

Using Lemma 2.2 and the bound (*ii*) in Lemma 2.3 with the symmetry of the scalar product in $\mathbb{R}^d$, we have:

$$\frac{1}{2}\int_\Gamma \alpha(v^{k+1} \cdot \nu) \, v^{k+1} \cdot \varphi \, ds \le \frac{c(\Omega)}{2} \left\|v^{k+1}\right\|_{0,\Omega}^{\frac{1}{2}} \left\|v^{k+1}\right\|_{1,\Omega}^{\frac{3}{2}} \left\|\varphi\right\|_{1,\Omega}.$$

With these bounds and $\left\|\varphi\right\|_{\mathbf{W}} = \left\|\varphi\right\|_{1,\Omega}$ for all $\varphi \in \mathbf{W}$, we get for all $k \in \mathbb{N}$ such that $(k+1)\delta t \le T$ for $d \le 3$:

$$\left\|\frac{v^{k+1} - v^k}{\delta t}\right\|_{\mathbf{W}'} \le \left\|f^{k+1}\right\|_{0,\Omega} + 2\mu_M \left\|D(v^{k+1})\right\|_{0,\Omega} + \frac{c(\Omega)}{2} \left\|v^{k+1}\right\|_{0,\Omega}^{\frac{1}{2}} \left\|v^{k+1}\right\|_{1,\Omega}^{\frac{3}{2}}.$$

From the convexity of the function $x :\longrightarrow x^{\frac{4}{3}}$ on $\mathbb{R}_+$, we have for all $a, b \ge 0$

$$\left(\frac{a+b}{2}\right)^{\frac{4}{3}} \le \frac{1}{2}\left(a^{\frac{4}{3}} + b^{\frac{4}{3}}\right), \qquad \text{and thus} \qquad (a+b)^{\frac{4}{3}} \le 2^{\frac{1}{3}}\left(a^{\frac{4}{3}} + b^{\frac{4}{3}}\right).$$

Then, we get using the first Korn inequality:

$$\begin{aligned}\left\|\frac{v^{k+1} - v^k}{\delta t}\right\|_{\mathbf{W}'}^{\frac{4}{3}} &\le c\left(\left\|f^{k+1}\right\|_{0,\Omega}^{\frac{4}{3}} + (2\mu_M)^{\frac{4}{3}} \left\|D(v^{k+1})\right\|_{0,\Omega}^{\frac{4}{3}} + c_1(\Omega) \left\|v^{k+1}\right\|_{0,\Omega}^{\frac{2}{3}} \left\|v^{k+1}\right\|_{1,\Omega}^{2}\right) \\ &\le c\left(\left\|f^{k+1}\right\|_{0,\Omega}^{\frac{4}{3}} + (2\mu_M)^{\frac{4}{3}} \left\|D(v^{k+1})\right\|_{0,\Omega}^{\frac{4}{3}} + c_2(\Omega) \left\|v^{k+1}\right\|_{0,\Omega}^{\frac{2}{3}} \left(\left\|v^{k+1}\right\|_{0,\Omega}^{2} + \left\|D(v^{k+1})\right\|_{0,\Omega}^{2}\right)\right).\end{aligned}$$

Hence, after multiplying by δt, summing up over $k = 0$ to n and using Hölder's inequality with the stability estimate (8) from Theorem 2.1, we get the desired bound. □

3. Global solvability of the Stokes problem with (OBC$_{|\Gamma}$) and uniqueness for $d \le 3$

In this section, we pass to the limit on the sequence of time consistent approximate solutions of our problem when the time step δt tends to zero using Aubin-Lions sequential compactness theorem [24, Chap. 1.5].

3.1. *Convergence of approximate solutions for $\delta t \to 0$*

For any integer $N \in \mathbb{N}$, we set $\delta t = \delta t_N = T/(N+1)$ such that $\delta t \to 0^+$ when $N \to +\infty$; further, we omit the index N for sake of simplicity in the notations and just say $\delta t \to 0$. Let us consider the partition of the interval $[0, T]$ formed by the points $t_k = k\delta t$ for $k = 0, \cdots, N+1$. By considering the solutions of the discrete scheme (6,7), let us construct the sequences of approximate solutions $(\overline{v}_{\delta t})$ and $(v_{\delta t})$ (for $\delta t = \delta t_N$, $N = 0, 1, 2, \cdots$) which are functions from $[0, T]$ into $\mathbf{W}$ defined by:

$$\begin{aligned}&\overline{v}_{\delta t}(0) = v^0 = v_0, \quad \overline{v}_{\delta t}(t) = v^{k+1}, \quad \forall t \in]t_k, t_{k+1}], \quad k = 0, \cdots, N \\ &v_{\delta t}(t) = \frac{t_{k+1} - t}{\delta t} v^k + \frac{t - t_k}{\delta t} v^{k+1}, \quad \forall t \in [t_k, t_{k+1}], \quad k = 0, \cdots, N.\end{aligned}$$

Thus, $\overline{v}_{\delta t}$ is piecewise constant on $[0, T]$, $v_{\delta t}$ is piecewise linear such that $v_{\delta t}(t_k) = v^k$ for all $k = 0, \cdots, N+1$, continuous in $[0, T]$, differentiable almost everywhere in $[0, T]$ and we have:

$$\frac{dv_{\delta t}}{dt}(t) = \frac{v^{k+1} - v^k}{\delta t}, \quad \forall t \in]t_k, t_{k+1}[, \quad k = 0, \cdots, N.$$

We also define the sequence $(\overline{f}_{\delta t})$ of functions in $L^2(0, T; \mathbf{L}^2(\Omega))$ by: $\overline{f}_{\delta t}(t) = f^{k+1}$ for all $t \in]t_k, t_{k+1}]$, $k = 0, \cdots, N$, which are piecewise constant on $[0, T]$.

Then, from the bounds (8) and (11), we have below the convergence results for these sequences.

Theorem 3.1 (**Convergence results for the sequences of approximate solutions** $(\overline{v}_{\delta t}), (v_{\delta t})$). *The sequences* $(\overline{v}_{\delta t})$ *and* $(v_{\delta t})$ *(*$\delta t = \delta t_N = T/(N+1)$, $N \in \mathbb{N}$*) are both bounded, independently of* δt*, in the space* $L^\infty(0,T;\mathbf{L}^2(\Omega)) \cap L^2(0,T;\mathbf{W})$ *and the sequence* $(\frac{dv_{\delta t}}{dt})$ *is bounded in the space* $L^{\frac{4}{3}}(0,T,\mathbf{W}')$ *for* $d \leq 3$*. We have also:*

$$\left\|v_{\delta t} - \overline{v}_{\delta t}\right\|_{L^2(0,T;\mathbf{L}^2(\Omega))} \longrightarrow 0 \quad \textit{when} \;\; \delta t \to 0.$$

Hence, there exists $v \in L^\infty(0,T;\mathbf{L}^2(\Omega)) \cap L^2(0,T;\mathbf{W})$ *with* $\frac{dv}{dt} \in L^{\frac{4}{3}}(0,T,\mathbf{W}')$ *and* $v \in C([0,T];\mathbf{W}')$*,* v *weakly continuous from* $[0,T]$ *into* $\mathbf{L}^2(\Omega)$ *such that, up to subsequences still denoted by the same notations, we have as* $\delta t = \delta t_N \to 0$ *when* $N \to +\infty$*:*

$$\begin{aligned}
&(i) \quad \overline{v}_{\delta t} \rightharpoonup v \quad \textit{and} \quad v_{\delta t} \rightharpoonup v \quad \textit{in} \;\; L^\infty(0,T;\mathbf{L}^2(\Omega)) \;\; \textit{weak-}\star\\
&(ii) \quad \overline{v}_{\delta t} \rightharpoonup v \quad \textit{and} \quad v_{\delta t} \rightharpoonup v \quad \textit{in} \;\; L^2(0,T;\mathbf{W}) \;\; \textit{weakly}\\
&(iii) \quad \frac{dv_{\delta t}}{dt} \rightharpoonup \frac{dv}{dt} \quad \textit{in} \;\; L^{\frac{4}{3}}(0,T;\mathbf{W}') \;\; \textit{weakly}\\
&(iv) \quad \overline{v}_{\delta t} \longrightarrow v \quad \textit{and} \quad v_{\delta t} \longrightarrow v \quad \textit{in} \;\; L^2(0,T;\mathbf{L}^2(\Omega)) \;\; \textit{strongly}\\
&(v) \quad (\overline{v}_{\delta t})_{|\Gamma} \longrightarrow v_{|\Gamma} \quad \textit{and} \quad (v_{\delta t})_{|\Gamma} \longrightarrow v_{|\Gamma} \quad \textit{in} \;\; L^2(0,T;\mathbf{L}^2(\Gamma)) \;\; \textit{strongly}\\
&(vi) \quad \overline{f}_{\delta t} \longrightarrow f \quad \textit{in} \;\; L^2(0,T;\mathbf{L}^2(\Omega)) \;\; \textit{strongly.}
\end{aligned}$$

Proof. The proof is standard from the works of [24, Chap. 4.1] and [26, Chap. III.4]. So, we just recall here the main arguments and for more details, we refer the reader to [16, Chap. 8.7] where a similar situation is studied for Navier-Stokes equations with homogeneous Dirichlet boundary conditions. Let us give some crucial hints in our case.

The boundedness of the sequences $(\overline{v}_{\delta t})$ and $(v_{\delta t})$ in the spaces $L^\infty(0,T;\mathbf{L}^2(\Omega))$ and $L^2(0,T;\mathbf{W})$ is an immediate consequence of their definitions using the estimates from (8). The boundedness of the sequence $(\frac{dv_{\delta t}}{dt})$ in the space $L^{\frac{4}{3}}(0,T,\mathbf{W}')$ comes immediately from the estimate (11) in Lemma 2.7.

Moreover, we easily get using the estimate (8):

$$\int_0^T \left\|v_{\delta t}(t) - \overline{v}_{\delta t}(t)\right\|_{0,\Omega}^2 dt = \frac{\delta t}{3} \sum_{k=0}^{N} \left\|v^{k+1} - v^k\right\|_{0,\Omega}^2 \leq \frac{C_0\, \delta t}{3}.$$

This shows that $(v_{\delta t} - \overline{v}_{\delta t}) \longrightarrow 0$ in $L^2(0,T;\mathbf{L}^2(\Omega))$ as $\delta t \to 0$ and thus $(\overline{v}_{\delta t})$ and $(v_{\delta t})$ have the same limit v.

Then, from the boundedness of the sequences $(\overline{v}_{\delta t})$ and $(v_{\delta t})$, the weak convergence follows, up to subsequences with possibly two extractions, from the Banach-Alaoglu theorem [9]. Since the weak convergence implies convergence in the sense of distributions in $\mathcal{D}'(]0,T[\times\Omega)$, the same weak weak limit is reached for each subsequence thanks to the uniqueness of the limit of a sequence in the sense of distributions. Since the time derivative operator is continuous in the distribution sense, the weak limit of $\frac{dv_{\delta t}}{dt}$ is necessarily $\frac{dv}{dt}$ in $L^{\frac{4}{3}}(0,T;\mathbf{W}')$.

Finally, the strong convergence of the sequence $(v_{\delta t})$ towards v in $L^2(0,T;\mathbf{L}^2(\Omega))$, and thus also the strong convergence of the sequence $(\overline{v}_{\delta t})$, follows from what precedes using Aubin-Lions compactness theorem [8, Theorem II.5.15] since the injection of $\mathbf{H}^1(\Omega)$ into $\mathbf{L}^2(\Omega)$ is compact.

Besides, from a trace inequality which holds for $d \leq 3$, we get with the Hölder inequality:

$$\int_0^T \left\|\overline{v}_{\delta t} - v\right\|_{0,\Gamma}^2 dt \leq c(\Omega)^2 \left(\int_0^T \left\|\overline{v}_{\delta t} - v\right\|_{0,\Omega}^2 dt\right)^{\frac{1}{2}} \left(\int_0^T \left\|\overline{v}_{\delta t} - v\right\|_{1,\Omega}^2 dt\right)^{\frac{1}{2}}.$$

The first integral in the right-hand side tends to zero with the strong convergence of $\overline{v}_{\delta t}$ in $L^2(0,T;\mathbf{L}^2(\Omega))$ whereas the second one remains bounded. Hence we have: $(\overline{v}_{\delta t})_{|\Gamma} \longrightarrow v_{|\Gamma}$ in $L^2(0,T;\mathbf{L}^2(\Gamma))$ strongly. □

3.2. Proof of Theorem 1.1

3.2.1. Passing to the limit in the approximate problem (6,7) and existence result for the problem (1,2)

By using the definitions of $(v_{\delta t})$, $(\overline{v}_{\delta t})$ and $(\overline{f}_{\delta t})$, we can interpret the weak form (7) of the scheme (6) with a time-dependent test function. With the regularity properties of $(v_{\delta t})$, $(\overline{v}_{\delta t})$ and $(\overline{f}_{\delta t})$ in Theorem 3.1, it makes sense to integrate in time over $[0,T]$ since we have $v \in C([0,T];\mathbf{W}')$ from [8, Proposition II.5.10]. Hence, we have for all $\psi \in C_c^\infty(]0,T[;\mathbf{W})$ since ψ is smooth in time:

$$\int_0^T \left\langle \frac{dv_{\delta t}}{dt}, \psi(t) \right\rangle_{\mathbf{W}',\mathbf{W}} dt + \int_0^T \int_\Omega 2\mu\, D(\overline{v}_{\delta t}) : D(\psi)\, dx\, dt + \frac{1}{2} \int_0^T \int_\Gamma \alpha(\overline{v}_{\delta t} \cdot v)\, \overline{v}_{\delta t} \cdot \psi\, ds\, dt = \int_0^T \int_\Omega \overline{f}_{\delta t}(t) \cdot \psi\, dx\, dt.$$

Now, using the convergence results in Theorem 3.1, we can pass to the limit of each term in the above equation as $\delta t \to 0$. The convergence is immediate in the linear terms using the weak limits. Let us deal with the non linear term which is non standard and requires compactness. More precisely, we have with the triangular inequality

$$\begin{aligned} &\left| \int_0^T \int_\Gamma (\alpha(\overline{v}_{\delta t} \cdot v)\, \overline{v}_{\delta t} \cdot \psi(t) - \alpha(v \cdot v)\, v \cdot \psi(t))\, ds\, dt \right| \\ &\qquad \le \int_0^T \int_\Gamma \left| \alpha(v \cdot v)\, (\overline{v}_{\delta t} - v) \cdot \psi(t) \right| ds\, dt + \int_0^T \int_\Gamma \left| \left(\alpha(\overline{v}_{\delta t} \cdot v) - \alpha(v \cdot v)\right)\, \overline{v}_{\delta t} \cdot \psi(t) \right| ds\, dt. \end{aligned}$$

Let us prove that each term in the right-hand side of this inequality tends to zero. For the first term, we use the continuity of $\alpha(.)$ from Lemma 2.2 and the estimate (i) in Lemma 2.3. Then, by using Hölder's inequality with $\left\|\psi(.)\right\|_{1,\Omega} \in L^\infty(]0,T[)$, it yields:

$$\int_0^T \int_\Gamma \left| \alpha(v \cdot v)\, (\overline{v}_{\delta t} - v) \cdot \psi(t) \right| ds\, dt \le c(\Omega) \left\|\psi\right\|_{L^\infty(0,T;\mathbf{H}^1(\Omega))} \left\|v\right\|_{L^2(0,T;\mathbf{H}^1(\Omega))} \left\|\overline{v}_{\delta t} - v\right\|_{L^2(0,T;\mathbf{L}^2(\Omega))} \left\|\overline{v}_{\delta t} - v\right\|_{L^2(0,T;\mathbf{H}^1(\Omega))}.$$

It shows that this term tends to zero since $\left\|\overline{v}_{\delta t} - v\right\|_{L^2(0,T;\mathbf{L}^2(\Omega))} \to 0$ thanks to the strong convergence of $\overline{v}_{\delta t}$ in $L^2(0,T;\mathbf{L}^2(\Omega))$ from Theorem 3.1, the other norms being bounded. For the second term, we use the Lipschitz continuity of $\alpha(.)$ from Lemma 2.2 and the estimate (i) in Lemma 2.3; we get

$$\int_0^T \int_\Gamma \left| \left(\alpha(\overline{v}_{\delta t} \cdot v) - \alpha(v \cdot v)\right)\, \overline{v}_{\delta t} \cdot \psi(t) \right| ds\, dt \le c(\Omega) \int_0^T \left\|\overline{v}_{\delta t} - v\right\|_{0,\Omega}^{\frac{1}{2}} \left\|\overline{v}_{\delta t} - v\right\|_{1,\Omega}^{\frac{1}{2}} \left\|\overline{v}_{\delta t}\right\|_{1,\Omega} \left\|\psi(t)\right\|_{1,\Omega} dt.$$

Then with Hölder's inequality and similarly as for the previous term, this term also tends to zero for the same arguments. Hence, it follows that

$$\left| \int_0^T \int_\Gamma \alpha(\overline{v}_{\delta t} \cdot v)\, \overline{v}_{\delta t} \cdot \psi(t)\, ds\, dt - \int_0^T \int_\Gamma \alpha(v \cdot v)\, v \cdot \psi(t)\, ds\, dt \right| \longrightarrow 0 \quad \text{when } \delta t \to 0.$$

Since, for $\psi(t) \in \mathbf{H}^1(\Omega)$ we have the trace $\psi(t)_{|\Gamma}$ in $\mathbf{L}^4(\Gamma)$ for $d \le 3$, this implies that: $(\alpha(\overline{v}_{\delta t} \cdot v)\, \overline{v}_{\delta t})_{|\Gamma} \longrightarrow \alpha(v \cdot v)\, v_{|\Gamma}$ in $L^2(0,T;\mathbf{L}^{\frac{4}{3}}(\Gamma))$ weakly. Hence, we can pass above to the limit as $\delta t \to 0$ and it yields for all $\psi \in C_c^\infty(]0,T[;\mathbf{W})$:

$$\int_0^T \left\langle \frac{dv}{dt}, \psi(t) \right\rangle_{\mathbf{W}',\mathbf{W}} dt + \int_0^T \int_\Omega 2\mu\, D(v) : D(\psi(t))\, dx\, dt + \frac{1}{2} \int_0^T \int_\Gamma \alpha(v \cdot v)\, v \cdot \psi(t)\, ds\, dt = \int_0^T \int_\Omega f(t) \cdot \psi(t)\, dx\, dt. \tag{12}$$

This shows that $v \in L^\infty(0,T;\mathbf{L}^2(\Omega)) \cap L^2(0,T;\mathbf{W})$, with $\dfrac{dv}{dt} \in L^{\frac{4}{3}}(0,T,\mathbf{W}')$ and $v \in C([0,T];\mathbf{W}')$, is a weak solution to the nonlinear Stokes problem (1,2) for $d \le 3$ and we have $v(0) = v_0$ in $\mathbf{W}'$ and weakly in $\mathbf{L}^2(\Omega)$.

3.2.2. Energy inequality

To obtain the energy inequality, the usual technique consists in taking the lower limit as $\delta t \to 0$ of the energy estimate of the discrete scheme using the fact that the norm is lower semi-continuous for the weak topology in a Banach space which is a consequence of the Banach-Steinhaus theorem; see [9].

By choosing the test function $\varphi = \delta t\, v^{n+1} \in \mathbf{W}$ in (7), we get through classical calculations:

$$\frac{1}{2} \left\|v^{n+1}\right\|_{0,\Omega}^2 - \frac{1}{2} \left\|v^n\right\|_{0,\Omega}^2 + \frac{1}{2} \left\|v^{n+1} - v^n\right\|_{0,\Omega}^2 + 2\delta t \left\| \sqrt{\mu}\, D(v^{n+1}) \right\|_{0,\Omega}^2 + \frac{1}{2}\, \delta t \left\| \sqrt{\alpha(v^{n+1} \cdot v)}\, v^{n+1} \right\|_{0,\Gamma}^2 = \delta t \int_\Omega f^{n+1} \cdot v^{n+1}\, dx.$$

For any $t \in [0, T]$, let us choose the sequence of mesh steps defined by $\delta t = \delta t_N = t/(N+1)$ for $N \in \mathbb{N}$. Then, after summing up the above equality for $n = 0$ to N and dropping the positive term $\left\|v^{n+1} - v^n\right\|_{0,\Omega}^2$, we get the inequality:

$$\frac{1}{2}\left\|v^{N+1}\right\|_{0,\Omega}^2 + 2\sum_{n=0}^{N} \delta t \left\| \sqrt{\mu}\, D(v^{n+1})\right\|_{0,\Omega}^2 + \frac{1}{2}\sum_{n=0}^{N} \delta t \left\| \sqrt{\alpha(v^{n+1}\cdot\nu)}\, v^{n+1}\right\|_{0,\Gamma}^2 \leq \frac{1}{2}\left\|v_0\right\|_{0,\Omega}^2 + \sum_{n=0}^{N} \delta t \int_{\Omega} f^{n+1}\cdot v^{n+1}\, dx.$$

Now using the definition of f^{n+1} and the sequence $(\overline{v}_{\delta t})$, we can interpret the previous inequality as below:

$$\frac{1}{2}\left\|\overline{v}_{\delta t}(t)\right\|_{0,\Omega}^2 + 2\int_0^t \left\| \sqrt{\mu}\, D(\overline{v}_{\delta t})\right\|_{0,\Omega}^2 d\tau + \frac{1}{2}\int_0^t \left\| \sqrt{\alpha(\overline{v}_{\delta t}\cdot\nu)}\, \overline{v}_{\delta t}\right\|_{0,\Gamma}^2 d\tau \leq \frac{1}{2}\left\|v_0\right\|_{0,\Omega}^2 + \int_0^t \int_{\Omega} f(\tau)\cdot \overline{v}_{\delta t}\, dx\, d\tau.$$

Then, using the weak convergence results stated above in Theorem 3.1, we can take the lower limit of the previous inequality which gives using the lower weak semi-continuity of the norm in a Banach space:

$$\frac{1}{2}\left\|v(t)\right\|_{0,\Omega}^2 + 2\int_0^t \left\| \sqrt{\mu}\, D(v)\right\|_{0,\Omega}^2 d\tau + \frac{1}{2}\int_0^t \left\| \sqrt{\alpha(v\cdot\nu)}\, v\right\|_{0,\Gamma}^2 d\tau \leq \frac{1}{2}\left\|v_0\right\|_{0,\Omega}^2 + \int_0^t \int_{\Omega} f(\tau)\cdot v(\tau)\, dx\, d\tau.$$

This is the desired energy inequality (5) and for more details, we refer the reader to the proof in [8, Chap. IV.1.9] for a similar situation.

3.2.3. Existence and regularity of the pressure field

The recovering of the pressure field in Ω is carried out in two steps. The first step is classical by using De Rham's theorem in $\mathbf{H}^{-1}(\Omega)$ at each time $t \in [0, T]$. We follow with minor modifications the detailed proof given in [8, Chap. IV.1.10] for the Navier-Stokes equations with homogeneous Dirichlet boundary condition. Hence, it gives the existence of $p_0 \in W^{-1,\infty}(]0, T[; L_0^2(\Omega))$ such that the Stokes equation is satisfied in $(0, T) \times \Omega$ in the sense of distributions, *i.e.* we have:

$$\frac{\partial v}{\partial t} - \operatorname{div}(2\mu\, D(v)) + \nabla p_0 = f \quad \text{in } \mathcal{D}'(]0, T[\times\Omega).$$

Moreover, the domain Ω being connected, the pressure field p_0 is unique as soon as the velocity field v is determined.

Second, we proceed similarly as in the proof of Theorem 2.5 for the stationary Stokes problem by using an *ad hoc* divergence-free extension in $\mathbf{W}$ of any function in $\mathbf{H}^{\frac{1}{2}}(\Gamma)$. Then, by testing the weak problem with any test function $\psi \in C_c^\infty(]0, T[; \mathbf{W})$ and taking account of the previous equation, we get the time-dependent function $C_\Gamma \in L^2(]0, T[)$ such that the open boundary condition (2) is satisfied in the weak sense of $L^2(0, T; \mathbf{H}^{-\frac{1}{2}}(\Gamma))$ with the pressure field $p = p_0 + C_\Gamma \in W^{-1,\infty}(]0, T[; L^2(\Omega))$.

This completes the proof of Theorem 1.1.

Remark 1 (On the uniqueness for the Stokes problem with $(\mathrm{OBC}_{|\Gamma})$ for $d = 2$). *In dimension $d = 2$, we can show using a refined trace estimate in Lemma 2.7 that the velocity time derivative $\frac{dv}{dt}$ only belongs to the space $L^{2-\epsilon}(0, T; \mathbf{W}')$ for any $0 < \epsilon < 1$. This is due to the kinetic energy term on the boundary and, roughly speaking, because $H^1(\Omega)$ is only embedded in $L^q(\Omega)$ for all $2 \leq q < +\infty$ and not in $L^\infty(\Omega)$, for $d = 2$. It follows that the uniqueness in 2-D, except for the linear case when $\alpha(.) = 0$, is not* a priori *ensured for this class of weak solutions.*

Nevertheless, if v is a little bit more regular such that $\frac{dv}{dt} \in L^2(0, T; \mathbf{W}')$, then the solution $v \in C\left([0, T]; \mathbf{L}^2(\Omega)\right)$ is unique and does satisfy the so-called energy equality. But the existence of such a solution is not guaranteed and needs a deeper analysis of this case.

References

[1] Ph. Angot, Analysis of singular perturbations on the Brinkman problem for fictitious domain models of viscous flows, Math. Meth. in the Appl. Sci., **22**(16) (1999), 1395–1412.

[2] Ph. Angot, A fictitious domain model for the Stokes/Brinkman problem with jump embedded boundary conditions, C. R. Math. Acad. Sci. Paris, Série I **348**(11-12) (2010), 697–702.

[3] Ph. Angot, On the well-posed coupling between free fluid and porous viscous flows, Applied Mathematics Letters, **24**(6) (2011), 803–810.

[4] Ph. Angot, F. Boyer and F. Hubert, Asymptotic and numerical modelling of flows in fractured porous media, Math. Model. Numer. Anal., **43**(2) (2009), 239–275.

[5] PH. ANGOT AND R. CHEAYTOU, Vector penalty-projection methods for incompressible fluid flows with open boundary conditions, in *Algoritmy 2012*, (A. Handlovičová et al., Eds), Slovak University of Technology in Bratislava, Publishing House of STU, Bratislava, (2012) 219–229.

[6] PH. ANGOT, J. CIHLÁR AND M. FEISTAUER, On the discretization and iterative solvers for viscous incompressible flow, in *Numerical Modelling in Continuum Mechanics*, (M. Feistauer et al., Eds), Vol. 1, Matfyzpress Publishing House, Charles Univ. Prague, (1997) 112–123.

[7] PH. ANGOT AND P. FABRIE, Convergence results for the vector penalty-projection and two-step artificial compressibility methods, Discrete Contin. Dyn. Syst., Ser. B, **17**(5) (2012), 1383–1405.

[8] F. BOYER AND P. FABRIE, *Eléments d'Analyse pour l'Étude de quelques Modèles d'Écoulements de Fluides Visqueux Incompressibles*, Mathématiques & Applications, **52**, Springer-Verlag, Berlin, 2006.

[9] H. BREZIS, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer-Verlag, New York, 2011.

[10] C.-H. BRUNEAU AND P. FABRIE, Effective downstream boundary conditions for incompressible Navier-Stokes equation, Int. J. Numer. Meth. in Fluids, **19** (1994), 693–705.

[11] C.-H. BRUNEAU AND P. FABRIE, New efficient boundary conditions for incompressible Navier-Stokes equation: a well-posedness result, Math. Model. Numer. Anal., **30**(7) (1996), 815–840.

[12] A.J. CHORIN AND J. MARSDEN, *A Mathematical Introduction to Fluid Mechanics*, Springer-Verlag, New York, 1992.

[13] R. COMOLET, *Mécanique expérimentale des fluides - Vol. II: Dynamique des fluides réels, turbomachines*, Masson, Paris, 3rd ed. 1982.

[14] C. CONCA, F. MURAT AND O. PIRONNEAU, Stokes and Navier-Stokes equations with boundary conditions involving the pressure, Japan J. Math., **20**(2) (1994).

[15] C. CONCA, C. PARES, O. PIRONNEAU AND M. THIRIET, Navier-Stokes equations with imposed pressure and velocity fluxes, Int. J. Numer. Meth. Fluids, **20** (1995), 267–287.

[16] M. FEISTAUER, *Mathematical methods in fluid dynamics*, Pitman Monographs and Surveys in Pure and Appl. Math. **67**, Longman Scientific & Technical and J. Wiley & Sons, New York, 1993.

[17] D. GILBARG AND N. TRUDINGER, *Elliptic Partial Differential Equations of Second Order*, 2nd ed., Springer-Verlag, New York, 1983.

[18] V. GIRAULT AND P.A. RAVIART, *Finite Element Methods for the Navier-Stokes Equations*, Springer Series in Comput. Math. **5**, Springer-Verlag (Berlin), 2nd ed. 1986.

[19] P.M. GRESHO AND R.L. SANI, Résumé and remarks on the open boundary condition minisymposium, Int. J. Numer. Meth. Fluids, **18** (1994), 983–1008.

[20] J. HEYWOOD, R. RANNACHER AND S. TUREK, Artificial boundaries and flux and pressure conditions for the incompressible Navier-Stokes equations, Int. J. Numer. Meth. Fluids, **22** (1996), 325–352.

[21] K. KHADRA, PH. ANGOT, S. PARNEIX AND J.-P. CALTAGIRONE, Fictitious domain approach for numerical modelling of Navier-Stokes equations, Int. J. Numer. Meth. in Fluids, **34**(8) (2000), 651–684.

[22] S. KRAČMAR AND J. NEUSTUPA, A weak solvability of a steady variational inequality of the Navier-Stokes type with mixed boundary conditions, Nonlin. Anal., **47** (2001), 4169–4180.

[23] P. KUČERA AND Z. SKALÁK, Local solutions to the Navier-Stokes with mixed boundary conditions, Acta Appl. Math., **54**(3) (1998), 275–288.

[24] J.-L. LIONS, *Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires*, Dunod & Gauthier-Villars, Paris, 1969.

[25] R. RANNACHER, Numerical analysis of the Navier-Stokes equations, Appl. of Math., **38**(4-5) (1993), 361–380.

[26] R. TEMAM, *Navier-Stokes Equations. Theory and Numerical Analysis*, Studies in Mathematics and its Applications **2**, North-Holland Publishing, Amsterdam, 4th ed. 1986.